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A STOCKPILING PROBLEM:
MATHEMATICAL TREATMENT

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SUMMARY

A problem proposed by A. Marshall in an internal working paper is solved in one general case and discussed in another.

A STOCKPILING PROBLEM: MATHEMATICAL TREATMENT

J. M. Danskin

§1. Introduction.

A. Marshall, in an internal memorandum, has proposed various stockpiling problems of which the present problem is the simplest. Indeed, a complete solution is obtained only in a special case, which unfortunately is not the case of greatest interest. For this point, see §2. A partial solution, that is, necessary conditions on the solution if there is one (if there is one it is unique), is given for the general case in §5. These conditions must be the starting point for further researches in the general case.

The problem is formulated in §2, discussed in §3. The results are given in §4, the necessary conditions in §5. An interesting general mathematical theorem, perhaps new, formulated by the present writer for L_1 and generalized and proved in any Banach space by I. Glicksberg, will be found in §7. In §8 we use the results of §7 and a compactness argument to prove the existence of a solution in Case I. The remainder of the paper is occupied with technical details.

§2. The problem.

The problem is to maximize the functional

$$\Gamma(y) = \int_0^\infty F\,[x(t)]\,v(t)dt$$

where

$$(1) \qquad x(t) = \bar{x} + \int_0^t H\,[y(\tau)e^{a\tau}]\,d\tau$$

and

$$(2) \qquad \int_0^\infty y(t)dt = C \quad .$$

Here F and H are strictly concave and strictly increasing, $H(u) \leq M$ for all u; C and M are constants; $v(t)$ is decreasing and $\int_0^\infty r(t)dt = 1$; $v(t) > 0$; $\bar{x} > 0$. The functions y are assumed nonnegative. A measurable nonnegative function y satisfying (2) is said to be in the class Ξ. A member of Ξ is called **admissible**.

§3. Discussion of the problem.

This problem arose as follows: it is desired to stockpile a certain good for a certain contingency, the latter happening only once. The probability that the contingency happens on $(t, t+dt)$ is given by $v(t)dt$. Production does not depend linearly on expenditure, but rather in a concave way; i.e. marginal production decreases with increasing expenditure. If z is the time rate of expenditure, the rate of production is given by $H(z)$. The utility of a stockpile x is given by $F(x)$. Again the marginal utility decreases with increasing

stockpile size; F is concave. It is desired to expend C units of resources, counted at present value, so as to maximise expected utility. Money accumulates at an interest rate a compounded continuously.

As proposed by Marshall, we understand that we intend to spend at time t at a (present value) rate $y(t)$. Thus actual expenditure at time t is at the rate $y(t)e^{at}$, and actual production $H\left[y(t)e^{at}\right]$. If the initial stockpile is $\bar{x}$, the stockpile at time t is given by (1).

An objection to Marshall's formulation may be made as follows. As both F and H are increasing, the problem is equivalent to the problem of minimising C, considered as a function of y, for constant $\Gamma(y)$. But it is clear that we will not spend C. The contingency will happen at some time t_0 and then we have

$$\int_{t_0}^{\infty} y(t)dt$$

(present value) resources left. It is a little unclear as to what it means to minimise C. If, however, we insert the factor

$$u(t) = \int_{t}^{\infty} v(\tau)d\tau \,, \tag{3}$$

the problem makes more sense. $u(t)$ is the probability that at time t the contingency has not as yet happened. Thus, if we plan to spend at time t at the (present value) rate $y(t)$, we will spend at an expected rate $y(t)u(t)$. Thus what we should minimise is the <u>expected present value of the cost</u>.

We should minimize

$$\int_0^\infty y(t)u(t)dt \quad .$$

at

Hence I propose that the problem be changed to read: maximize $\Gamma(y)$, where $x(t)$ is given by (1), subject to the side condition

$$\int_0^\infty y(t)u(t)dt = C \quad ,$$

where $u(t)$ is given by (3).

In the case that $v(t) = \beta e^{-\beta t}$, of interest to Marshall, this problem reduces to the problem proposed by him. Unfortunately, at present, it reduces to a type we are so far unable to solve. I shall discuss this point in the next section.

The discussion in sections §4-§10 is all based on the problem as formulated by Marshall.

§4. <u>The results. Discussion.</u>

<u>Case I. The function v satisfies</u>

$$(4) \qquad \frac{v(t)}{\int_t^\infty v(t)dt} \geq \beta > \alpha$$

<u>for all</u> t .

In this case, there is an interval $(0, t_o)$ and a positive constant k, such that

(i) if $t < t_o$, $y_o(t)$ satisfies the equation

$$(5) \qquad H'\left[y_o(t)e^{\alpha t}\right] = \frac{k}{e^{\alpha t}\int_t^{\infty} F'\left[x_o(\tau)\right] v(\tau)d\tau}$$

(ii) if $t \geq t_o$, $y_o(t) = 0$.

k and t_o are determined by the side condition (2) and by the equation

$$(5a) \qquad H'(0) = \frac{k}{e^{\alpha t_o}\int_{t_o}^{\infty} F'\left[x_o(\tau)\right] v(\tau)d\tau} .$$

The problem of solving these rather complicated equations is not part of our problem here. Further progress will probably require special assumptions on F and H.

We observe that the solution $y_o(t)$ in this case is steadily decreasing and that it is continuous everywhere, including the point t_o . This is proved in §9.

Case II. The function $v(t)$ satisfies

$$(6) \qquad \frac{v(t)}{\int_t^\infty v(\tau)\,d\tau} < \beta + \alpha$$

for all t.

Practically nothing is known in general for Case II. It can be proved that in this case the stockpile $x(t)$ approaches infinity, so that $y_0(t)$ — if there is a solution $y_0(t)$ — must be positive at arbitrarily large values of t.

The proposed modification given in §3 of Marshall's problem falls into this category when $v(t) = \beta e^{-\beta t}$. Then the problem comes down to one with an interest rate of $\alpha + \beta$ and a contingency rate of β. Thus (6) is always fulfilled.

No results at all are known for the general case, except the following: the solution is always unique, and if there is a solution it must satisfy the necessary conditions of §5. Except in Case I, the existence of a solution has not been proved.

§5. Necessary conditions

In this section we derive conditions, in the form of inequalities, which y_0 satisfies if it yields a maximum to $\Gamma(y)$ in the class Ξ of admissible functions.

Let y_0 be the maximizing function, and suppose that y_1 is any other element of Ξ. Let λ be on the unit interval. Write $y_\lambda(t) = (1-\lambda)y_0(t) + \lambda y_1(t)$. Put

$$x_\lambda(t) = \bar{x} + \int_0^t H\left[y_\lambda(\tau)e^{\alpha\tau}\right]d\tau .$$

Put

$$\phi_{y_1}(\lambda) = \int_0^\infty F\left[x_\lambda(t)\right] v(t)dt .$$

Evidently a necessary condition that y_o yield a maximum is that

$$\frac{\partial}{\partial\lambda}\left[\phi_{y_1}(\lambda)\right]_{\lambda=0} \leq 0$$

for all $y_1 \in \Xi$. On differentiating and reversing the order of integration, this yields:

$$\int_0^\infty H'\left[y_o(t)e^{\alpha t}\right] Q_{y_o}(t)\left[y_o(t) - y_1(t)\right] dt \geq 0$$

for all $y_1 \in \Xi$, where we have written

$$Q_{y_1}(t) = e^{\alpha t} \int_t^\infty F'\left[x_1(\tau)\right] v(\tau)d\tau .$$

We now apply the Gibbs-Neyman-Pearson lemma (see [1] , p. 289).

We get the

Necessary conditions: If y_o maximizes $\Gamma(y)$ in the class Ξ, there exists a constant κ such that, for almost all t,

(7) (i) if $y_o(t) > 0$, then $H'\left[y_o(t)e^{\alpha t}\right] = \frac{k}{Q_{y_o}(t)}$;

(ii) if $y_o(t) = 0$, then $H'\left[y_o(t)e^{\alpha t}\right] \leq \frac{k}{Q_{y_o}(t)}$.

§6. Additional necessary conditions holding in Case I.

If the contingency is likely to occur fairly soon, i.e. Case I, we are able to obtain an estimate of k.* With the aid of this we prove in this section that there exist numbers t^* and Y_o such that if y_o yields a maximum to $\Gamma(y)$ in Ξ then $y_o(t) \leq Y_o$ throughout and $y_o(t) = 0$ for $t \geq t^*$.

First we turn to $Q_{y_1}(1)$. Since $H(u) \leq M$ for all n by hypothesis, then for any $y_1 \in \Xi$ we have

$$(8) \qquad x_1(t) \leq \bar{x} + Mt \quad ,$$

for all t. Hence

$$(9) \qquad Q_{y_1}(1) \geq e^{a} \int_1^{\infty} F'(\bar{x}+Mt)v(t)dt \quad .$$

The quantity on the right of (9) evidently does not depend on $y_1 \in \Xi$.

Next we prove that under the conditions of Case I the quantity $Q_{y_1}(t)$ is decreasing in t for every $y_1 \in \Xi$. Observe that

$$\frac{F'[x_1(t)]v(t)}{\int_t^{\infty} F'[x_1(\tau)]v(\tau)d\tau} \geq \frac{v(t)}{\int_t^{\infty} v(\tau)d\tau} \geq \beta$$

throughout. Hence

$$\frac{d}{dt}\left[\log Q_{y_1}(t)\right] = a - \frac{F'[x_1(t)]v(t)}{\int_t^{\infty} F'[x_1(\tau)]v(\tau)d\tau} \leq a - \beta \quad .$$

---

* which holds also in the general case.

It follows that $Q_{y_1}(t)$ is decreasing in t, and also that

$$Q_{y_1}(t) \leq Q_{y_1}(0)e^{(\alpha-\beta)t} .$$

On observing that evidently

$$Q_{y_1}(0) < F'(0)$$

we get

$$Q_{y_1}(t) \leq F'(0)e^{(\alpha-\beta)t} . \tag{10}$$

The right side of (10) is evidently independent of $y_1 \in \Xi$.

Now we can obtain a lower bound for k. It follows from (2) that there is a set of positive measure on $0 \leq t \leq 1$ on which $y_o(t) < 2C$. Hence for some t on $[0,1]$ we have

$$H'(2Ce^{\alpha}) < H'\left[y_o(t)e^{\alpha t}\right] \leq \frac{k}{Q_{y_o}(t)} .$$

Hence

$$k > H'(2Ce^{\alpha})Q_{y_o}(t) \geq H'(2Ce^{\alpha})e^{(t-1)\alpha}Q_{y_o}(1) \geq H'(2Ce^{\alpha}) \int_1^{\infty} F'(\bar{x}+Mt)v(t)\,dt . \tag{11}$$

Write

$$\kappa_0 = H'(2Ce^{\alpha}) \int_1^{\infty} F'(\bar{x}+Mt)v(t)dt \ .$$

Thus we have proved that $\kappa \geq \kappa_0 > 0$, where κ_0 is an absolute constant.

Observe that this estimate for κ does not hold only in Case I, but in general.

Now put

$$t^{*} = \frac{1}{\beta-\alpha} \log \frac{H'(0)F'(0)}{\kappa_0} \ .$$

Obviously $t^{*} > 0$. Making use of (10), it follows easily that

$$\lambda_{y_0}(t) < \frac{\kappa_0}{H'(0)} \ ,$$

for all $t \geq t^{*}$. Now suppose $y_0(t) > 0$ for some $t > t^{*}$, where t does not lie in the set of measure zero for which (7) does not hold. Then

$$\lambda_{y_0}(t) = \frac{\kappa}{H'[y_0(t)e^{\alpha t}]} \ .$$

Accordingly we would have

$$\frac{\kappa}{H'[y_0(t)e^{\alpha t}]} < \frac{\kappa_0}{H'(0)} \ .$$

But $\kappa \geq \kappa_0$ and $H'[y_0(t)e^{\alpha t}] \leq H'(0)$, and so we have a contradiction. Hence if $t \geq t^{*}$ then $y_0(t) = 0$.

We now establish the existence of an absolute constant Y_0 such that $y_0(t) \leq Y_0$ throughout. Take Y_0 so that

$$(12) \qquad H'(Y_0) = \frac{k_0}{F'(0)} .$$

Since $k_0 < H'(0)F'(0)$ there is obviously such as Y_0. Suppose that at a point t satisfying (7) we have $y_0(t) > Y_0$. Then $y_0(t) > 0$ and so

$$H'\left[y_0(t)e^{\alpha t}\right] = \frac{k}{Q_{y_1}(t)} .$$

Then

$$H'\left[y_0(t)\right] > H'\left[y_0(t)e^{\alpha t}\right] = \frac{k}{Q_{y_1}(t)} \geq \frac{k_0}{Q_{y_1}(t)} > \frac{k_0}{F'(0)} .$$

It follows that $y_0(t) < Y_0$, a contradiction. This completes the proof of the assertions in the first paragraph of this section.

§7. A theorem on upper semicontinuity.

The result of this section is needed in the proof given in §8 of the existence of a solution in Case I.

Theorem: Any strongly continuous concave functional J on a Banach space is upper semicontinuous with respect to weak convergence.

Proof*: Let $\{u_\delta\}_{\delta \in \Delta}$ be a directed set of elements of the Banach space, converging weakly to an element u_o. Put

$$\mu = \limsup_{\delta \in \Delta} J(u_\delta) .$$

Let $\varepsilon > 0$. There is a cofinal subset Δ_1 of Δ such that if $\delta \in \Delta_1$ then

$$J(u_\delta) > \mu - \frac{\varepsilon}{2} .$$

Let D be the strong convex closure of Δ_1. Then by Mazur's theorem (see [3] or [5], p. 52[illegible]), D is weakly closed. Hence $u_o \in D$. Accordingly, u_o is the strong limit of a directed set of finite convex combinations of elements of Δ_1. Hence there is a finite convex combination $\sum \lambda_i u_{\delta_i}$ of elements of Δ_1 such that

$$| J(u_o) - J(\sum \lambda_i u_{\delta_i}) | < \frac{\varepsilon}{2} .$$

It follows that

$$J(u_o) > J(\sum \lambda_i u_{\delta_i}) - \frac{\varepsilon}{2} \geq \sum \lambda_i J(u_{\delta_i}) - \frac{\varepsilon}{2} > \mu - \varepsilon .$$

As ε is arbitrary, it follows that

$$J(u_o) \geq \mu ,$$

as required. This completes the proof.

* For which I am indebted to I. Glicksberg

We add a remark. We use this theorem only in the case where the Banach space in L_1 and the functional is an integral. In this case the result follows from a theorem of McShane [2]. We give the theorem in the present form because of its intrinsic interest and generality.

§8. Existence of a solution in Case I.

In this section we will prove that in Case I there exists a solution to the problem enunciated in §2. Discussion of the solution will be found in §4. Other cases are touched on in §4 and §10.

First we consider a restricted problem. Let $\Xi_Y^{t'}$ denote the class of functions in Ξ which satisfy $y(t) \leq Y$ throughout and $y(t) = 0$ for $t > t'$. Take as the topology of $\Xi_Y^{t'}$ the topology of the Banach space L_1. For any fixed t, the real-valued functional

$$x(t) = \bar{x} + \int_0^t H\left[y(\tau)e^{\alpha\tau}\right]d\tau$$

is upper semicontinuous with respect to weak convergence on $\Xi_Y^{t'}$. This follows from the results of §7 and from the boundedness of $H'(u)$. The factor $e^{\alpha\tau}$ introduces no difficulty; we need only refer to the definition of weak convergence and the finiteness of the interval $(0,t)$. It follows that the functional

$$\Gamma(y) = \int_0^\infty F\left[x(t)\right]v(t)dt$$

is upper semicontinuous with respect to weak convergence on $\Xi_Y^{t'}$. As $\Xi_Y^{t'}$ is

compact (see [4], p. 136) in the weak topology, there exists a maximum in $\Xi_Y^{t'}$.

Observe next that since F and H are strictly concave, there is a unique maximum in $\Xi_Y^{t'}$. Let y_0 be the unique maximum lying in $\Xi_{Y_0}^{t^*}$, where t^* and Y_0 are the quantities defined in §6.

Let now $t' > t^*$, $Y > Y_0$. Let y_0' be the unique maximum lying in $\Xi_Y^{t'}$. By an argument similar to that of §6, y_0' must satisfy the following condition:

<u>There exists a constant</u> k, <u>such that, for almost all</u> t,

$$\begin{cases} \text{(i) } \underline{\text{if}} \ \ y_0'(t) = Y \ , \ \underline{\text{then}} \ \ H'\left[y_0'(t)e^{at}\right] \geqq \dfrac{k}{Q_{y_0'}(t)} \ ; \\ \text{(ii) } \underline{\text{if}} \ \ 0 < y_0'(t) < Y, \ \underline{\text{then}} \ \ H'\left[y_0'(t)e^{at}\right] = \dfrac{k}{Q_{y_0'}(t)} \ ; \\ \text{(iii) } \underline{\text{if}} \ \ y_0'(t) = 0 \ , \ \underline{\text{then}} \ \ H'\left[y_0'(t)e^{at}\right] \leqq \dfrac{k}{Q_{y_0'}(t)} \ . \end{cases}$$

Assume for the moment that $Y \geqq 2C$. Then, as in §6, we obtain a set on $0 \leqq t \leqq 1$ of positive measure on which $y_0(t) < 2C \leqq Y$. As the estimate of §6 for $Q_{y_0'}(1)$ clearly is not affected by the restriction to $\Xi_Y^{t'}$, the same k_0 will serve as served there. Now take Y_0 as before ((12)). It follows much the same as before that $y_0'(t) \leqq Y_0$ throughout. In the same way as before, it follows also that $y_0'(t) = 0$ for $t \geqq t^*$. Hence y_0' lies in $\Xi_{Y_0}^{t^*}$. Hence it maximizes in $\Xi_{Y_0}^{t^*}$. Therefore it is identical with the unique maximum in $\Xi_{Y_0}^{t^*}$. Obviously we may drop the assumption $Y \geqq 2C$. We have proved the following result:

<u>Lemma</u>: *If $t \geq t^*$ and $Y \geq Y_0$, the unique maximum in Ξ_Y^t is given by the unique maximum y_0 of $\Xi_{Y_0}^{t^*}$.*

Now we obtain an absolute upper bound for $\Gamma(y)$ for $y \in \Xi_Y$ in Case I. Recalling (8), we have

$$(13) \qquad \Gamma(y) < \int_0^\infty F\,(\bar{x}+Mt)v(t)dt < F'(0)\int_0^\infty (\bar{x}+Mt)v(t)dt.$$

It follows easily from (4) that $u(t) = \int_t^\infty v(\tau)d\tau < e^{-\beta t}$; accordingly the right side of (13) converges. Hence it is the desired absolute upper bound.

Suppose that for some $y \in \Xi$, $\Gamma(y) > \Gamma(y_0)$. Let y^{t_0} be gotten from y by putting $y(t) = 0$ for $t > t_0$. Then if t_0 is sufficiently large, $\Gamma(y^{t_0}) > \Gamma(y_0)$. This follows from the bound (13) on Γ. Fix such a t_0. Let Y be large. The measure of the set of points $E_Y^{t_0}$ on $(0,t_0)$ for which $y(t) > Y$ may be made arbitrarily small by taking Y sufficiently large. Recalling the formula for $x(t)$ and the fact that $H(u) \leq M$ for all u, it follows that if Y is sufficiently large and

$$y_Y^{t_0}(t) = \begin{Bmatrix} Y & \text{for } t \in E_Y^{t_0} \\ y^{t_0}(t) & \text{for } t \notin E_Y^{t_0} \end{Bmatrix},$$

then

$$\Gamma(y_Y^{t_0}) > \Gamma(y_0) .$$

Now put

$$y_1(t) = \lambda y_Y^{t_o}(t) \quad ,$$

where

$$\lambda = \frac{C}{\int_0^{t_o} y_Y^{t_o}(t)dt} \geqq 1 \, .$$

Then $y_1 \in \Xi_{\lambda Y}^{t_o}$, and $\Gamma(y_1) \geqq \Gamma(y_Y^{t_o}) > \Gamma(y_o)$. This is a contradiction. Accordingly, for every $y \in \Xi$, $\Gamma(y) \leqq \Gamma(y_o)$.

Hence y_o yields a maximum to $\Gamma(y)$ for $y \in \Xi$. This completes the proof of existence in the Case I.

§9. <u>The solution in Case</u> I.

The form of the solution in Case I is given in §4. We shall prove those statements here.

Suppose first that t_1 and t_2 are two points at which (7) holds. Take $t_2 > t_1$. Recall that $Q_{y_o}(t)$ is strictly decreasing. Suppose $y_o(t_2) > 0$. <u>Then</u> $y_o(t_1) > 0$. For suppose $y_o(t_1) = 0$. Then

$$H'[y_o(t_1)e^{\alpha t_1}] = H'(0) > H'[y_o(t_2)e^{\alpha t_2}] = \frac{k}{Q_{y_o}(t_2)} > \frac{k}{Q_{y_o}(t_1)} \quad ,$$

a contradiction.

Recalling that $y_o(t) = 0$ for $t \geq t^*$, the existence of a t_o as asserted in §4 follows.

$y_o(t)$ is obviously continuous for $t \neq t_o$. Let us check continuity at $t = t_o$. Let

$$\ell = \limsup_{t \to t_o -} y_o(t)e^{\alpha t} .$$

The limit is taken through points at which (7) holds. Then $H'(\ell) = \frac{k}{Q_{y_o}(t_o)}$. But on taking the limit from the right, we get $H'(0) = \frac{k}{Q_{y_o}(t_o)}$. Hence $\ell = 0$ as desired. We have also proved by this argument that k and t_o satisfy (5a).

§10. The solution in Case II.

As we have said before, little is known in this case. We can prove that if there is a solution the stockpile is unbounded.

In §6 we saw that the quantity k_o did not depend on the Case (Case I) under consideration. However, the existence of a bound Y_o and an upper limit t^* for the lines used did depend on the assumptions of Case I. What was needed there was that $Q_{y_o}(t)$ should be decreasing. Let us suppose the stockpile bounded. Then it follows from (6) that $Q_{y_o}(t)$ is strictly increasing after a certain point. From (7) it follows that $y_o(t)e^{\alpha t}$ is increasing also. This is a contradiction.

The reader is referred to H. Kahn* for various conjectures in this case with specialized F and H and v.

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